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Instantons versus the low-temperature expansion†

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Abstract. We examine the O(2) nonlinear σ model at finite temperature in one dimension. The relationship between the low-temperature expansion (the analogue to the $1/N$ expansion) and instanton calculation is clarified. We show how the seemingly non-perturbative (in $\hbar$) instanton calculation can be re-expressed as a power series in $\hbar$. Further, our calculations suggest that unstable instantons also play a role in path integral calculations.

1. Introduction

It is now accepted that perturbative expansions (in $\hbar$) for local quantum field theories are not necessarily adequate, and non-perturbative approaches have been developed. Two such methods are semiclassical instanton methods (Coleman 1977, 't Hooft 1976) and the $1/N$ expansion ('t Hooft 1974, Witten 1979, Coleman 1980).

In the instanton method (see Coleman 1977, 't Hooft 1976 and references therein) the number of fields, N, is fixed and the Euclidean functional integrals approximated in the $\hbar \to 0$ limit. Thus, an expansion is developed around the finite action classical solutions to the Euclidean equations of motion. On the other hand, the $1/N$ expansion corresponds to holding $\hbar$ fixed and performing the Minkowski functional integrals by saddle-point approximation in the $N \to \infty$ limit.

There has been considerable discussion about the equivalence or otherwise of the two approaches. Recently, Jevicki (1979) has proposed a program for analysing the equivalence of the two approximations in theories like the linear σ models and nonlinear σ models (NLSM). However, even for such simple models there are difficulties. Here, we apply Jevicki's method to the simpler case of finite-temperature quantum mechanics. Before explaining what we hope to learn from this example we briefly summarise Jevicki's method.

Jevicki observed that in such models the introduction of auxiliary fields can provide a framework for instanton calculations, as well as the $1/N$ expansion. For example, in the two-dimensional O(N) invariant NSLM the (Euclidean) partition function

$$Z = \int \prod_{i=1}^{N} [\mathrm{d}\varphi_i][\delta(\varphi_1^2 - Nf^2)] \exp -\frac{1}{2\hbar} \int (\partial_\mu \varphi_i)^2 \, \mathrm{d}^2 x \tag{1.1}$$

† This is a revised and extended version of ICTP/79-80/19.

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can be re-expressed as

$$Z = \int [\mathrm{d}\lambda] \prod_1^N [\mathrm{d}\varphi_i] \exp -\frac{1}{2\hbar} \int \mathrm{d}^2x [(\partial_\mu \varphi_i)^2 + \mathrm{i}\lambda (\varphi_i^2 - Nf^2)] \tag{1.2}$$

$$= \int [\mathrm{d}\lambda][\det(-\partial^2 + \mathrm{i}\lambda)]^{-N/2} \exp \frac{\mathrm{i}Nf^2}{2\hbar} \int \mathrm{d}^2x\lambda. \tag{1.3}$$

Z of equation (1.3) can be evaluated in two different ways. For instance, for each multi-instanton solution, $\varphi^{(n)}$, there is a value, $\lambda^{(n)}$, of the auxiliary field such that $\det(-\partial^2 + \mathrm{i}\lambda^{(n)})$ is zero. Thus, we can evaluate Z by deforming the 'contour' in λ 'space' to enclose the zeros of $\det(-\partial^2 + \mathrm{i}\lambda)$ associated with the classical solutions $\lambda^{(n)}$. Using Cauchy's theorem for the degrees of freedom for which the zeros occur and the Gaussian approximation for the others in the $\hbar \to 0$, fixed N limit we should reproduce the instanton calculation†.

Alternatively, equation (1.3) can be rewritten as

$$Z = \int [\mathrm{d}\lambda] \exp -N\mathfrak{A}[\lambda] \tag{1.4}$$

and evaluated in the $N \to \infty$, fixed $\hbar$ limit by deforming the λ 'contour' to pass through the saddle point, λ_0, of the 'effective action'

$$\mathfrak{A}[\lambda] = \frac{\mathrm{i}f^2}{2\hbar} \int \mathrm{d}^2x\lambda - \tfrac{1}{2}\mathrm{Tr} \ln(-\partial^2 + \mathrm{i}\lambda). \tag{1.5}$$

This is the $1/N$ expansion.

Thus, the equivalence (or not) of the instanton and $1/N$ approximation depends on the ability to deform contours and the commutativity of the limits $\hbar \to 0$, $N \to \infty$. There is no reason to expect them to give identical results, but the above formalism gives a common framework in which to examine each method.

Jevicki's approach seems so intuitively correct that it might be wondered why we should want to demonstrate it in such a simple model (the one-dimensional, finite temperature NLSM) and what we hope to learn. However, the argument presented by Jevicki and summarised above is oversimplified. For example, there is not necessarily a one-to-one correspondence between instanton configurations $\varphi^{(n)}$ and auxiliary field solutions $\lambda^{(n)}$ (which may not distinguish between instantons and anti-instantons). Furthermore the heuristic claim that all relevant poles come from classical solutions needs to be examined in detail to check its validity.

2. Nonlinear σ models in one dimension

In order to understand the relationship between instanton and $1/N$ expansions better we wish to concentrate on theories for which the $N \to \infty$ and $\hbar \to 0$ limits commute. This is the case whenever the instanton calculations are exact. Theories for which the semiclassical approximation is exact include quantum mechanics of free particles moving on a manifold, $\mathfrak{M}$, of a simple Lie group (Schulman 1968, Dowker 1970, 1971). To have instantons we must consider finite-temperature quantum statistical mechanics (i.e. all Euclidean fields $\phi(\tau)$ have periodicity with period $\beta = (kT)^{-1}$). To permit stable

† We only have stable instantons for $N = 3$, but the generalisation to more realistic models is straightforward.

instantons we require $\Pi_1(\mathfrak{M})$ to be non-trivial. In practice, we are forced to consider the O(2) nonlinear σ model in one dimension at finite temperature. Here the manifold is S^1 and $\Pi_1(S^1) = \mathbb{Z}$. (The O(4) model is exactly solvable but does not have stable instantons†.)

Since we are forced to consider the O(2) NLS model how can we do a sensible $1/N$ expansion for $N = 2$? The answer is not to do so, but to re-examine the motivation for the $1/N$ expansion. For the $1/N$ expansion to be possible the numerator and denominator of the path integral under consideration have to vary equally rapidly for large N. In our case we have the parameter, $\beta = (kT)^{-1}$, which becomes large as $T \to 0$. Fixing N and taking the $\beta \to \infty$ limit the numerator and denominator again vary equally rapidly, permitting a low-temperature expansion. This is combinatorially similar to a large N expansion. Thus, for our O(2) model we compare the (exact) instanton calculation to a $1/\beta$ expansion, using Jevicki's programme. For the remainder of this section we shall discuss the model and its exact solution.

The O(2) NLSM has a single complex field φ satisfying

$$|\varphi|^2 = f^2 \tag{2.1}$$

and classical action

$$A = \tfrac{1}{2}\int_0^\beta \mathrm{d}\tau |\dot{\varphi}|^2. \tag{2.2}$$

It is sufficient to evaluate the two-point function‡

$$\langle \varphi(\tau)\varphi^*(0)\rangle = \frac{\int [\mathrm{d}\varphi][\mathrm{d}\varphi^*][\delta(|\varphi|^2 - f^2)]\varphi(\tau)\varphi^*(0)\exp(-A/\hbar)}{\int [\mathrm{d}\varphi][\mathrm{d}\varphi^*][\delta(|\varphi|^2 - f^2)]\exp(-A/\hbar)}. \tag{2.3}$$

Parametrising φ as

$$\varphi = \rho \exp \mathrm{i}\tilde{\chi} \qquad \rho > 0 \qquad -\infty < \tilde{\chi} < \infty \tag{2.4}$$

equation (2.3) can be rewritten as§

$$\langle \varphi(\tau)\varphi^*(0)\rangle = \frac{f^2 \int [\mathrm{d}\chi] E[\tilde{\chi}] \exp \mathrm{i}(\tilde{\chi}(\tau) - \tilde{\chi}(0))}{\int [\mathrm{d}\chi] E[\tilde{\chi}]} \tag{2.5}$$

where $\chi = \tilde{\chi}$ (mod 2π) and

$$E[\tilde{\chi}] = \exp -\frac{f^2}{2\hbar}\int_0^\beta \mathrm{d}\tau\, \dot{\tilde{\chi}}^2. \tag{2.6}$$

All closed periodic paths must be considered in equation (2.5). We first restrict ourselves to those paths for which

$$\chi(0) = \chi(\beta) = \chi_0. \tag{2.7}$$

† Only the O(2) model has stable instantons. This is analogous to the two-dimensional ce where only the O(3) model has stable instantons. However, in the one-dimensional case generalising to other manifolds (like CP^{N-1}) does not help. We lose the infinite-connectness, the identification with a Lie group, or both.

‡ We wish to keep track of factors of $\hbar$ for the reasons indicated in the introduction. To bring the formalism into accord with conventional instanton calculations, where classical solutions are $\hbar$ independent, we have not replaced β by $\beta\hbar$.

§ In general it is not permissible to replace $\dot{\varphi}_1^2 + \dot{\varphi}_2^2$ by $\dot{\rho}^2 + \rho^2\dot{\tilde{\chi}}^2$ in path integrals (Edwards and Gulyaev 1964). However, when dealing with periodic paths at fixed ρ all that happens is a change of normalisation, with no effect on the ratio (2.5).

The classical paths are

$$\tilde{\chi}_n(\tau)=\omega_n\tau+\chi_0 \qquad n\in\mathbb{Z} \tag{2.8}$$

where

$$\omega_n=2\pi n/\beta. \tag{2.9}$$

These are the instanton solutions, with classical action

$$A_n=\tfrac{1}{2}f^2\beta\omega_n^2. \tag{2.10}$$

The path $\tilde{\chi}_n$ is the classical path with winding number n. Taking into account quantum fluctuations, a general path of winding number n has the form

$$\tilde{\chi}(\tau)=\tilde{\chi}_n(\tau)+\eta(\tau) \tag{2.11}$$

with

$$\eta(0)=\eta(\beta)=0. \tag{2.12}$$

The contribution of these paths to the partition function, the denominator of (2.5) is

$$Z_n=(\exp-A_n/\hbar)\int[\mathrm{d}\eta]E[\eta] \tag{2.13}$$

with (2.12) imposed.

Similarly, the numerator of equation (2.5) acquires a contribution from these paths of the form

$$T_n=[\exp(\mathrm{i}\omega_n\tau-A_n/\hbar)]\int[\mathrm{d}\eta]E[\eta]\exp\mathrm{i}\eta(\tau). \tag{2.14}$$

Both (2.13) and (2.14) are solvable.

To obtain the two-point function from (2.13) and (2.14) we must sum over all homotopy classes. There is the well known phase ambiguity between contributions from paths belonging to different homotopy classes. The only acceptable phases in this case are

$$\gamma_\theta(n)=\exp\mathrm{i}n\theta \qquad -\pi\leqslant\theta\leqslant\pi \tag{2.15}$$

giving two-point functions†

$$\langle\varphi(\tau)\varphi^*(0)\rangle_\theta=f^2\Big(\sum_n\gamma_\theta(n)T_n\Big)\Big(\sum_n\gamma_\theta(n)Z_n\Big)^{-1}. \tag{2.16}$$

These sums can be expressed in terms of Jacobi θ_3 functions as

$$\langle\varphi(\tau)\varphi^*(0)\rangle_\theta=\frac{f^2\theta_3(\pi\tau/\beta+\frac{1}{2}\theta|2\pi\mathrm{i}f^2/\beta\hbar)}{\theta_3(\frac{1}{2}\theta|2\pi\mathrm{i}f^2/\beta\hbar)}\exp\left[-\frac{\beta\hbar}{2f^2}\frac{\tau'}{\beta}\left(1-\frac{\tau'}{\beta}\right)\right] \tag{2.17}$$

where $\tau'=\tau\ (\mathrm{mod}\ \beta)$.

From equation (2.17) we see that at high temperature ($\beta\to0$)

$$\langle\varphi(\tau)\varphi^*(0)\rangle_\theta\xrightarrow[\beta\to0]{}f^2 \tag{2.18}$$

† All paths are taken into account by integrating out the final degree of freedom $0<\chi_0\leqslant2\pi$. This final integration gives $\langle\varphi(\tau)\rangle=0$. Thus $\langle\varphi(\tau)\varphi^*(0)\rangle$ is connected.

independent of θ i.e. the instanton effects vanish. This is not surprising since instantons can only have significant effects if they can act over long periods of time (Polyakov 1977), and periodicity in time with vanishing period β prevents this happening.

Of more interest is the low-temperature limit ($\beta\to\infty$). We note that, using the duality property of θ_3 functions,

$$\langle\varphi(\tau)\varphi^*(0)\rangle_\theta=\frac{f^2\theta_3((\hbar/2\mathrm{i}f^2)(\tau+\beta\theta/2\pi)|-\beta\hbar/2\pi\mathrm{i}f^2)}{\theta_3(\hbar\beta\theta/4\pi\mathrm{i}f^2|-\beta\hbar/2\pi\mathrm{i}f^2)}\exp\left[-\frac{\tau'\hbar}{2f^2}\left(1+\frac{\theta}{\pi}\right)\right] \tag{2.19}$$

$$\xrightarrow[\beta\to\infty]{} f^2\exp\left[-\frac{\tau\hbar}{2f^2}\left(1+\frac{\theta}{\pi}\right)\right]. \tag{2.20}$$

That is, we have a θ-dependent correlation 'mass' $m(\theta)$ given by

$$m(\theta)=\frac{\hbar}{2f^2}\left(1+\frac{\theta}{\pi}\right). \tag{2.21}$$

It is in this result that we shall be primarily interested.

3. The auxiliary field

In this section we consider the auxiliary field formalism and indicate how the exact results of the previous section can be obtained by contour integration in the λ plane. We examine the role played by the classical solutions in this integration. The partition function can be written as

$$Z=\int[\mathrm{d}\varphi][\mathrm{d}\varphi^*][\mathrm{d}\lambda]\exp-\frac{1}{2\hbar}\int_0^\beta \mathrm{d}\tau[|\dot\varphi|^2+\mathrm{i}\lambda(|\varphi|^2-f^2)]. \tag{3.1}$$

The classical equations of motion

$$(-\partial^2+\mathrm{i}\lambda)\varphi(\tau)=0=|\varphi|^2-f^2 \tag{3.2}$$

have solutions

$$\varphi^{(n)}(\tau)=f\exp\mathrm{i}\tilde\chi_n(\tau) \tag{3.3}$$

and

$$\lambda^{(n)}(\tau)=\mathrm{i}\omega_n^2=\mathrm{i}(2\pi n/\beta)^2. \tag{3.4}$$

For $n\neq0$, $\lambda^{(n)}=\lambda^{(-n)}$, showing a two-fold degeneracy of the classical solutions. The $\varphi-\varphi^*-\lambda$ manifold is simply connected. Thus, if we perform a semiclassical approximation for (3.1) the phases from different saddle-point contributions are fixed.

In order to reproduce the phase ambiguity of (2.15), i.e. θ vacua, (3.1) is replaced by

$$Z_\theta=\int[\mathrm{d}\varphi][\mathrm{d}\varphi^*][\mathrm{d}\lambda]\exp-\frac{1}{\hbar}\int_0^\beta \mathrm{d}\tau[L_\theta+\tfrac12\mathrm{i}\lambda(|\varphi|^2-f^2)] \tag{3.5}$$

where

$$L_\theta=\tfrac12|\dot\varphi|^2-\frac{\hbar\theta}{2\pi f^2}(\dot\varphi\varphi^*-\dot\varphi^*\varphi). \tag{3.6}$$

The equations of motion are unchanged, but the λ solutions become

$$\lambda^{(n)}(\theta)=\Omega_n^2\equiv\omega_n\left(\omega_n-\frac{\mathrm{i}\theta\hbar}{\pi f^2}\right). \tag{3.7}$$

For $\theta\neq 0$ the solutions are now non-degenerate.

Integrating over φ and φ^* we have $(G(\theta)=-\partial^2+\mathrm{i}\lambda-\theta\hbar\partial/\pi f^2)$

$$\begin{aligned} Z_\theta &= \int[\mathrm{d}\lambda]\det G(\theta)\exp\frac{\mathrm{i}f^2}{2\hbar}\int \mathrm{d}\tau\lambda \\ &= \int\prod_n \mathrm{d}\lambda_n \det(\Omega^2+\mathrm{i}\Lambda)^{-1}\exp \mathrm{i}f^2\lambda_0/2\hbar \end{aligned} \tag{3.8}$$

in terms of the Fourier components

$$\lambda_q=\lambda_{-q}^*=\beta^{-1}\int_0^\beta \mathrm{d}\tau\,\lambda(\tau)\exp \mathrm{i}\omega_n\tau \tag{3.9}$$

with

$$\Omega^2=\mathrm{diag}(\ldots,\Omega_{-p}^2,\ldots,\Omega_{-2}^2,\Omega_{-1}^2,\Omega_1^2,\Omega_2^2,\ldots,\Omega_p^2,\ldots) \tag{3.10}$$

and

$$(\Lambda)_{ij}=\lambda_{i-j}. \tag{3.11}$$

Expressed as Fourier components (3.7) becomes

$$\lambda_p^{(n)}=\delta_{p,0}\lambda^{(n)}(\theta)=\delta_{p,0}\Omega_n^2. \tag{3.12}$$

Consider the surfaces of zeros passing through the solution $\lambda^{(n)}$. For small λ_p, $p\neq 0$ (denoted λ') we can expand $\det(\Omega^2+\mathrm{i}\Lambda)$ as

$$\det(\Omega^2+\mathrm{i}\Lambda)=\left(\prod_{p\neq n}(\Omega_p^2-\Omega_n^2)\right)(\mathrm{i}\lambda_0-\mathrm{i}\Lambda^{(n)}(\lambda')) \tag{3.13}$$

where the surfaces of zeros $\Lambda^{(n)}(\lambda')$ have the form

$$\mathrm{i}\Lambda^{(n)}(\lambda')=\Omega_n^2+\sum_{q\neq 0}\frac{(\mathrm{i}\lambda_q)(\mathrm{i}\lambda_{-q})}{\Omega_{q+n}^2-\Omega_n^2}+\mathrm{O}(\lambda'^3). \tag{3.14}$$

This gives a contribution to Z_θ of

$$Z_{\theta,n}=\int\frac{\mathrm{d}\lambda_0\Pi_{p\neq 0}\,\mathrm{d}\lambda_p}{\Pi_{p\neq n}(\Omega_p^2-\Omega_n^2)}\frac{\exp \mathrm{i}f^2\beta\lambda_0/2\hbar}{(\lambda_0-\Lambda^{(n)}(\lambda'))}. \tag{3.15}$$

The situation is described graphically in figure 1. Two of the axes describe Re λ_0, Im λ_0 and the third λ'. The curve of interest describes $\Lambda^{(n)}(\lambda')$ of (3.14) as λ' varies. We integrate over λ_0 first. From (3.5) the deformation of the contour is permissible. By

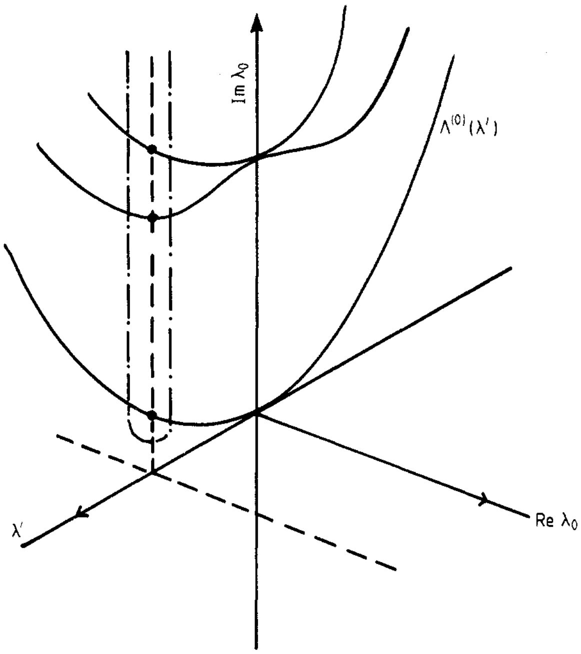

Figure 1. The surfaces of zeros of det $G(\theta)$ for $\theta=0$. The curves in the Im $\lambda_0-\lambda'$ plane denote the solution $\Lambda^{(0)}(\lambda')$ (lower curve) and $\Lambda_{\pm}^{(1)}(\lambda')$ (upper two curves). For fixed λ' (broken line) the contour in the λ_0 plane is folded up around the $\Lambda^{(n)}(\lambda')$ (chain curve). For $\theta\neq 0$ the classical solutions move off the imaginary axis for $n\neq 0$, and the curves are no longer coincident at the classical solutions.

Cauchy's theorem we have

$$Z_{\theta,n}=\frac{2\pi}{\Pi_{p\neq n}(\Omega_p^2-\Omega_n^2)}\int\prod_{p\neq 0}\mathrm{d}\lambda_p\ \exp \mathrm{i}f^2\beta\Lambda^{(n)}(\lambda')/2\hbar. \tag{3.16}$$

In the semiclassical approximation ($\hbar\to 0$) we obtain

$$\begin{aligned}Z_\theta&=\sum_n Z_{\theta,n}\propto\sum_n \exp(n\theta-A_n/\hbar)\\ &=\theta_3\left(\frac{\theta}{2}\middle|\frac{2\pi\mathrm{i}f^2}{\beta\hbar}\right).\end{aligned} \tag{3.17}$$

What is not obvious, in this formalism, is why the semiclassical approximation (3.17) is exact, as we know from § 2.

The case for $\theta=0$ deserves further study. On the one hand we can get the result from (3.17). On the other hand we can make a direct computation for $\theta=0$. However, the situation is different from the $\theta=0$ case since the surfaces of zeros $\Lambda_{\pm}^{(n)}(\lambda')$ are coincident in pairs at the classical solution $\lambda^{(n)}$, $n\neq 0$. Because of these double zeros the behaviour of $\det(-\partial^2+\mathrm{i}\lambda)$ is more complicated. It can be seen that the surfaces are not harmonic in the λ_{2n} direction. As before, we fix λ', deform the λ_0 contour to pass around the two poles now present. In the semiclassical approximation non-Gaussian contributions interfere to give a Gaussian result.

To summarise, we see that we need knowledge of the surfaces of zeros of $\det(-\partial^2+\mathrm{i}\lambda)$ in the vicinity of the classical solutions. It is *not* true that all relevant poles come from the classical solutions, as claimed in Jevicki (1979).

4. The low-temperature expansion

In the last section we showed how the instanton calculation can be reproduced in the auxiliary field formalism by contour integration. Here we show how the same results can be achieved by large β saddle-point calculations, technically analogous to the $1/N$ expansion.

Defining

$$E_1(\theta)=\exp\left(\frac{\mathrm{i}f^2}{2\hbar}\int \mathrm{d}\tau\,\lambda-\operatorname{Tr}\ln G(\theta)\right) \tag{4.1}$$

we already have, from (3.8), that

$$Z_\theta=\int [\mathrm{d}\lambda]E_1(\theta). \tag{4.2}$$

Let us first consider $\theta=0$. In Fourier components

$$\det G(0)=\det(\Omega^2+\mathrm{i}\Lambda)$$
$$=4\Big(\prod_{p\neq 0}\omega_p^2\Big)\sinh^2\tfrac{1}{2}\beta\sqrt{\mathrm{i}\lambda_0}\,\det[1+(\Omega^2+\mathrm{i}\lambda_0\mathbb{1})^{-1}(\Lambda-\lambda_0\mathbb{1})] \tag{4.3}$$

giving

$$Z=\int\frac{\Pi_p\,\mathrm{d}\lambda_p}{4\,\Pi_{p\neq 0}\,\omega_p^2}\exp[\mathrm{i}f^2\beta\lambda_0/2\hbar-\ln\sinh^2(\beta\sqrt{\mathrm{i}\lambda_0}/2)]$$
$$\times\exp-\operatorname{Tr}\ln[1+(\Omega^2+\mathrm{i}\lambda_0\mathbb{1})^{-1}(\Lambda-\lambda_0\mathbb{1})]. \tag{4.4}$$

Each term in the exponents in (4.5) is $\mathrm{O}(\beta)$. Thus, as $\beta\to\infty$, Z can be evaluated by saddle-point methods, expanding about the solutions to

$$\sqrt{\mathrm{i}\lambda_0}=\tfrac{1}{2}f^{-2}\hbar\coth(\beta\sqrt{\mathrm{i}\lambda_0}/2). \tag{4.5}$$

The most significant of these is $\lambda=\bar{\lambda}$, given by

$$\mathrm{i}\bar{\lambda}=f^{-4}\hbar^2+\mathrm{O}(\exp-c\beta). \tag{4.6}$$

Consider again the two-point function. This is

$$\langle\varphi(\tau)\varphi^*(0)\rangle=2\frac{\int[\mathrm{d}\lambda]\langle\tau|G(0)^{-1}|0\rangle E_1(0)}{\int[\mathrm{d}\lambda]E_1(0)}. \tag{4.7}$$

As $\beta\to\infty$ we obtain

$$\langle\varphi(\tau)\varphi^*(0)\rangle=2\beta^{-1}\sum_n\frac{\exp\mathrm{i}\omega_n\tau}{\omega_n^2+\hbar^2f^{-4}} \tag{4.8}$$

$$=\frac{f^2\cosh f^{-2}\hbar(\frac{1}{2}\beta-\tau)}{\sinh\frac{1}{2}f^{-2}\hbar\beta} \tag{4.9}$$

$$\sim f^2\exp-\hbar\tau/f^2. \tag{4.10}$$

Thus the large β saddle-point also gives rise to a mass gap (of $m=\hbar/f^2$ for $\theta=0$). However, the agreement with the exact case is only qualitative, the correlation mass being twice that of (2.21).

Before examining this discrepancy we stress that we have performed a large β saddle-point calculation, i.e. only the large β limit of (4.9) is reliable. However, had we performed a $1/N$ expansion and then set $N=2$ we would have obtained (4.10) for arbitrary β. For small β (4.10) gives the classical result

$$\langle\varphi(\tau)\varphi^*(0)\rangle\sim f^2 \tag{4.11}$$

showing the $1/N$ expansion to be accurate here for $N=2$†.

As to the discrepancy, we can only assume that the single saddle point is insufficient. Other extrema exist lying on the positive, imaginary λ_0 axis between the poles of the classical solutions $\lambda^{(n)}$, given by $\lambda_0=\mathrm{i}\eta^2$ where

$$\eta\beta=\sin^{-1}\frac{2\eta f^2\hbar}{\hbar^2+\eta^2 f^4}+(2n+1)\pi. \tag{4.12}$$

As $\beta\to\infty$ they coalesce to $\lambda_0=0$. Although the action for these solutions is zero (rather than negative, as for $\lambda=\bar{\lambda}$) the interplay between these extrema and the pole at $\lambda_0=0$ (the classical vacuum) is complicated, and we are unable to see in detail how the exact result is attained.

For $\theta\neq 0$ the calculation is similar, though more tedious. We have

$$\langle\varphi(\tau)\varphi^*(0)\rangle_\theta=\frac{2\int[\mathrm{d}\lambda]\langle\tau|G(\theta)^{-1}|0\rangle E_1(\theta)}{\int[\mathrm{d}\lambda]E_1(\theta)} \tag{4.13}$$

and the large β saddle point is

$$\mathrm{i}\bar{\lambda}(\theta)=f^{-4}\hbar^2(1-\theta^2/4\pi^2) \tag{4.14}$$

and (4.13) becomes

$$\langle\varphi(\tau)\varphi^*(0)\rangle_\theta\sim\exp-\tau f^{-2}\hbar(1-\theta/2\pi) \tag{4.15}$$

in the large β limit, i.e.

$$\langle\varphi(\tau)\varphi^*(0)\rangle_\theta=\langle\varphi(\tau)\varphi^*(0)\rangle_{\theta=0}\exp-\tau\theta\hbar/2\pi f^2 \tag{4.16}$$

as with the exact case, though $\bar{\lambda}(0)$ above gives the incorrect mass gap, as noted earlier.

Remembering that the example considered here was one of those quoted in Jevicki (1979) in support of the (large β saddle point)–(instanton) equivalence, this shows that the hopes of Jevicki are unfulfilled, because of the greater complexity than expected from heuristic manipulations.

5. Generalisations

In § 2 we observed that the manifestly non-$\hbar$-perturbative instanton calculation for the two-point function $(0\leqslant\tau\leqslant\beta)$

$$\langle\varphi(\tau)\varphi^*(0)\rangle=f^2\frac{\Sigma_n\exp(\mathrm{i}\omega_n\tau-A_n/\hbar)}{\Sigma_n\exp-A_n/\hbar}\exp\left[-\frac{\hbar\tau}{2f^2}\left(1-\frac{\tau}{\beta}\right)\right] \tag{5.1}$$

† In one dimension there is no critical temperature, and hence no critical value of N. This may explain the accuracy of the $1/N$ expansion for such small N. This would not necessarily be true in more complicated theories.

can be expressed perturbatively in $\hbar$ as $(0 \leqslant \tau \leqslant \beta)$

$$\langle \varphi(\tau)\varphi^*(0)\rangle = f^2 \frac{\Sigma_n \exp[(n-\frac{1}{2})\hbar\tau/f^2] \exp -[A_n\beta^2\tau\hbar/(2\pi f^2)^2]}{\Sigma_n \exp -A_n\beta^2\tau\hbar/(2\pi f^2)^2}. \tag{5.2}$$

This property is generally true for exactly solvable models. To see this consider the solvable case when the manifold of fields is that of an r-parameter semi-simple Lie group G. Let us consider the *Feynman propagator*, K, for moving from ω to ω' in time t on G. This is exactly solvable (Dowker 1970, 1971) by means of classical paths as

$$K = A \exp(\tfrac{1}{12}\mathrm{i}\hbar t R)(2\pi \mathrm{i} t)^{-r}\vartheta(t) \sum_j \exp(\mathrm{i} s_j^2/2\hbar t). \tag{5.3}$$

R is the scalar curvature of the manifold and A is expressible in terms of the roots of G. From our point of view the most important factor is the infinite sum, running over all classical paths, j, from ω to ω', of length s_j. Note that (apart from time ordering) there is no obvious difficulty in analytically continuing t to $-\mathrm{i}\tau$.

Reverting to the *Euclidean* functional integrals for one-dimensional field theory we see that we are only interested in closed geodesics traversed in Euclidean time β. Thus the basic sum is (with matrix metric m)

$$Z = \sum_j \exp(-s_j^2/2\beta\hbar) = \theta(0|\mathrm{i}m/\beta\hbar) \tag{5.4}$$

where θ is a multi-dimensional ϑ function† with matrix argument. We are only interested in how $\hbar$ arises in the arguments.

In a typical two-point function, $G(\tau)$, Z of (5.4) would be the denominator. The numerator is of generic form

$$T = \theta(m\omega(\tau)|\mathrm{i}m/\beta\hbar)g(\hbar, \tau) \tag{5.5}$$

where $g(\hbar, \tau)$ describes the Gaussian fluctuations, with a convergent expansion in powers of $\hbar$, and ω is an $\hbar$-independent column matrix. We are only interested in the way $\hbar$ occurs in the final form

$$G(\tau) = \frac{\theta(m\omega(\tau)|\mathrm{i}m/\beta\hbar)}{\theta(0|\mathrm{i}m/\beta\hbar)} g(\hbar, \tau). \tag{5.6}$$

As in the O(2) case of (2.17) both the numerator and the denominator of G, containing factors $\exp(-s^2/2\beta\hbar)$ term by term, have zero asymptotic series in $\hbar$. However, by the duality transformation of θ we can re-express G as

$$G(\tau) = \frac{\theta(\mathrm{i}\beta\hbar\omega(\tau)|\mathrm{i}\beta\hbar m^{-1})}{\theta(0|\mathrm{i}\beta\hbar m^{-1})} \exp(-\mathrm{i}\beta\hbar\tilde{\omega}m\omega/\pi)g(\hbar, \tau). \tag{5.7}$$

Comparing (5.6) with (5.7) we see that all non-perturbative terms $\mathrm{e}^{-A/\hbar}$ have been replaced by terms $\mathrm{e}^{-\beta\hbar}$. Thus, both numerator and denominator can be expressed in a power series in $\hbar$, as would be implied by $1/N$ expansions, or any expansion corresponding to a re-ordering of the Feynman series (for a model with no infinite renormalisation). It is conceivable that, whenever $1/N$ expansions and instanton

† If n (with elements n_j) and Z are N column matrices and T an $N \times N$ matrix

$$\theta(Z|T) = \sum_{n_j=-\infty}^{n_j=\infty} \exp(2\mathrm{i}\tilde{n}Z + \mathrm{i}\pi\tilde{n}Tn).$$

calculations give identical results, a similar re-expression of terms $O(e^{-aN})$ into terms $O(e^{-A/N})$ has taken place. The presence of terms $O(e^{-aN})$ should not necessarily imply the absence of an expansion in $1/N$.

Our final comment concerns the role of unstable instantons. For example, consider the one-dimensional Euclidean $O(N)$ NLS model, $N > 2$. The manifold is S^{N-1} and the closed periodic paths on S^{N-1} are great circles, traversed n times, $n \in \mathbb{Z}$. Thus, the classical paths are embeddings of the O(2) multi-instanton paths in $O(N)$†. Since S^{N-1} is simply connected there is no conserved, topological charge. Thus, all classical paths for $n \neq 0$ are *unstable*. However, for $N = 4$ we have an exactly solvable, quantum mechanical model. Hence the quantum mechanics of the O(4) model is entirely determined by the classical paths, whether stable or not. Thus, if analytic continuation from real to imaginary time is possible without encountering any singularities, the unstable instantons will dominate the Euclidean path integral.

Let us parametrise S^3 by the Weyl angles $\omega^i (i = 1, \ldots, 4)$ satisfying $\Sigma_i \omega^i = 0$, $-\pi \leqslant \omega^i \leqslant \pi$. The quantum mechanical sum over closed paths beginning and ending at $\omega^i = 0$ is (Dowker 1970, 1971)

$$Z = \sum_j \exp(\mathrm{i}s_j^2/2t) = \sum_{l_i=-\infty}^{l_i=+\infty} \exp(2\pi^2 \mathrm{i}\tilde{l}ml/t) \tag{5.8}$$

where l is the 3×1 column vector with elements l_i $(i = 1, 2, 3)$ and m is the positive definite matrix defined by

$$\tilde{l}ml = 8(l_1^2 + l_2^2 + l_3^2 + l_1 l_2 + l_2 l_3 + l_3 l_1). \tag{5.9}$$

There is no difficulty in continuing from $t \to -\mathrm{i}\tau$ in (5.8). This suggests that, for models where the instanton calculations are exact the unstable instantons have to be included in the calculation.

6. Conclusions

In this paper we have examined the exactly solvable O(2) NLS model at finite temperature. We have provided a detailed example of Jevicki's program for comparing different non-perturbative (in $\hbar$) approaches‡. In particular, the nature of the contour integration in the auxiliary field formalism (to reproduce the instanton results) has been clarified. We have seen that knowledge of the surfaces of zeros in the neighbourhood of the classical solutions is necessary. This is particularly so when the classical solutions are degenerate. Some equivalence of the contour integration to the large β saddle-point calculation is indicated. We have been unable to establish a direct equality although both calculations give a mass gap. However, the mass gap in the single saddle-point approximation is twice that of the exact result. Presumably the other saddle points play a role. We have only considered the two-point function in the paper because higher-order connected n-point Green functions are non-leading (in $1/N$ and $1/\beta$), and agreement is even less likely to occur.

† A similar situation occurs for the two-dimensional $O(N)$ NLS model. Only for $N = 3$ is there a conserved topological charge. For $N > 3$ embeddings of the O(3) solution in $O(N)$ provide unstable, classical solutions (Din and Zakrzewski 1980).

‡ The O(2) NLS model was briefly considered in Jevicki (1979).

For exact solutions, when all methods must be equivalent, we are able to see how seemingly non-perturbative (in $\hbar$) instanton calculations can be re-expressed as a power series in $\hbar$. This is due to the duality transformations of generalised θ_3 functions, in terms of which the solutions to exactly solvable models can be expressed. Also, there are indications that analyticity in time is straightforward for exactly solvable models. This suggests that, for such models, unstable instantons should be included in instanton calculations.

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Note added. A calculation by G Segal (communication from W Nahm) shows that analytic continuation in τ for the exactly solvable models is indeed as simple as we have suggested.

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